

Indian Institute of Information Technology Allahabad

Real Analysis (RA) - Quiz I
(Tentative Marking Scheme)

Program: B.Tech. (DA221)

Date: September 25, 2026

Full Marks: 15

Time: 14:40 - 15:20 IST

Instructions: Attempt all the questions. All the notations are standard and same as used in the classes and lecture notes.

1. Let $x_n \leq 0$ for every $n \in \mathbb{N}$. Suppose (x_n) is a convergent sequence such that $x_n \rightarrow l$. Show that $l \leq 0$. [3]

Solution. Suppose $l > 0$. Choose $\epsilon_0 > 0$ such that $0 < \epsilon_0 < l$. Let $\epsilon = l - \epsilon_0 > 0$. [1]

Then for $\epsilon = l - \epsilon_0 > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - l| < \epsilon = l - \epsilon_0$. [1]

$\implies x_n > \epsilon$ for all $n \geq N$. A contradiction. [1]

2. Let $x_0 \in \mathbb{Q}^c$. Show that there exists a sequence (x_n) in $\mathbb{Q}$ such that $x_n \rightarrow x_0$. [3]

Solution. Since rationals are dense in irrationals, $\forall n \in \mathbb{N}$, we get a sequence (x_n) in $\mathbb{Q}$ such that $x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$.

$\implies x_0 - \frac{1}{n} < x_n < x_0 + \frac{1}{n}$, that is, $\frac{1}{n} < x_n - x_0 < \frac{1}{n}$. [2]

By sandwich theorem, $x_n - x_0 \rightarrow 0$, that is, $x_n \rightarrow x_0$. [1]

3. Let $x_1 = \sqrt{2}$, and $x_n = \sqrt{2 + x_{n-1}}$ for $n > 1$. Show that the sequence (x_n) is convergent. Also find the limit of the sequence. [3]

Solution. By induction, we show that $x_n \leq 2$ for every n .

Clearly $x_1 \leq 2$. Let $x_n \leq 2$ for $n = k$. Then $x_{k+1} = \sqrt{2 + x_k} \leq \sqrt{2 + 2} = 2$. [1]

Thus $x_n \leq 2$ for every n .

Similarly, by induction, we show that x_n is increasing. Clearly $x_2 \geq x_1$.

Let $x_k \geq x_{k-1}$. Then $x_{k+1} - x_k = \sqrt{2 + x_k} - \sqrt{2 + x_{k-1}} = \frac{x_k - x_{k-1}}{\sqrt{2 + x_k} + \sqrt{2 + x_{k-1}}} \geq 0$.

Thus x_n is increasing sequence and bounded above by 2, so it will be convergent. [1]

Suppose $x_n \rightarrow l$. Then $l = \sqrt{2 + l}$. This implies $l = 2$. [1]

4. Discuss the convergence/divergence of the following series [6]

(i) $\sum_{n=1}^{\infty} \log(\frac{n+1}{n})$ (ii) $\sum_{n=1}^{\infty} \frac{1}{n!}$ (iii) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Solution (i) $S_n = \log(n+1)$. Since $S_n \rightarrow \infty$ as $n \rightarrow \infty$. So the series $\sum_{n=1}^{\infty} \log(\frac{n+1}{n})$ is divergent. [2]

(ii) Note that $\frac{1}{2^n} \geq \frac{1}{n!}$ for $n \geq 4$. Since $\sum \frac{1}{2^n}$ is convergent, by comparison test, the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ is also convergent. [2]

(iii) Since $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = (1 + \frac{1}{n})^n = e > 1$, by ratio test, the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent. [2]